Recall from the notes:

Let $a_1(x)$ be continuous on I.

Let y_1 be a solution to

$$y'' + a_1(x)y' + a_0(x)y = 0$$

on I where $y_1(x) \neq 0$ for all x in I.

Then,

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

will be another solution that is linearly independent with y_1

$\boxed{1(a)}$ We are given that $y_1 = x^4$ is a solution to

$$x^2y'' - 7xy' + 16y = 0$$

on $I = (0, \infty)$.

Note that $y_1 = x^4 \neq 0$ on $I = (0, \infty)$.

Divide by x^2 to get the equation

$$y'' \underbrace{- \frac{7}{x}}_{a_1(x) = -\frac{7}{x}} y' + \frac{16}{x^2} y = 0$$

Using our formula from class we get

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

$$= x^4 \int \frac{e^{-\int -\frac{7}{x} dx}}{(x^4)^2} dx$$

$$= x^4 \int \frac{e^{7\int \frac{1}{x} dx}}{x^8} dx$$

$$= x^4 \int \frac{e^{7\ln|x|}}{x^8}\,dx$$

$x > 0$ when x is in $I = (0, \infty)$ So, $|x| = x$

$$= x^4 \int \frac{e^{7\ln(x)}}{x^8}\,dx$$

$A\ln(B) = \ln(B^A)$

$$= x^4 \int \frac{e^{\ln(x^7)}}{x^8}\,dx$$

$e^{\ln(z)} = z$

$$= x^4 \int \frac{x^7}{x^8}\,dx$$

$$= x^4 \int \frac{1}{x}\,dx$$

$$= x^4 \ln|x|$$

$|x| = x$ since $x > 0$ when x is in $I = (0, \infty)$

$$= x^4 \ln(x)$$

Thus, $y_1 = x^4$ and $y_2 = x^4\ln(x)$ are two linearly independent solutions to $x^2y'' - 7xy' + 16y = 0$ on $I = (0, \infty)$. And the general solution to $x^2y'' - 7xy' + 16y = 0$ on $I = (0, \infty)$ is of the form $y = c_1y_1 + c_2y_2 = c_1x^4 + c_2x^4\ln(x)$

1(b) We are given that $y_1 = x^2$ is a solution to

$$x^2 y'' + 2xy' - 6y = 0$$

on $I = (0, \infty)$.

Note that $y_1 = x^2 \neq 0$ on $I = (0, \infty)$.

Divide by x^2 to get the equation

$$y'' + \underbrace{\frac{2}{x}}_{a_1(x) = \frac{2}{x}} y' - \frac{6}{x^2} y = 0$$

Using our formula from class we get

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)\,dx}}{y_1^2}\,dx$$

$$= x^2 \int \frac{e^{-\int \frac{2}{x}\,dx}}{(x^2)^2}\,dx$$

$$= x^2 \int \frac{e^{-2\int \frac{1}{x}\,dx}}{x^4}\,dx$$

$$= x^2 \int \frac{e^{-2\ln|x|}}{x^4}\,dx$$

$x > 0$ when x is in $I = (0, \infty)$ So, $|x| = x$

$$= x^2 \int \frac{e^{-2\ln(x)}}{x^4}\,dx$$

$A\ln(B) = \ln(B^A)$

$$= x^2 \int \frac{e^{\ln(x^{-2})}}{x^4}\,dx$$

$e^{\ln(z)} = z$

$$= x^2 \int \frac{x^{-2}}{x^4}\,dx$$

$$= x^2 \int x^{-6}\,dx$$

$$= x^2 \cdot \frac{x^{-6+1}}{-6+1}$$

$$= -\frac{1}{5}x^{-3}$$

$\boxed{\text{Thus, } y_1 = x^2 \text{ and } y_2 = -\frac{1}{5}x^{-3} \text{ are two linearly independent solutions to } x^2y'' + 2xy' - 6y = 0 \text{ on } I = (0,\infty). \text{ And the general solution to } x^2y''+2xy'-6y=0 \text{ on } I=(0,\infty) \text{ is of the form } y = c_1y_1 + c_2y_2 = c_1x^2 + c_2\left(-\frac{1}{5}x^{-3}\right)}$

$\boxed{1(c)}$ We are given that $y_1 = \ln(x)$ is a solution to

$$xy'' + y' = 0$$

on $I = (0, \infty)$.

Note that $y_1 = \ln(x) \neq 0$ on $I = (0, \infty)$.

Divide by x to get the equation

$$y'' + \underbrace{\frac{1}{x}}_{a_1 = \frac{1}{x}} y' = 0$$

Using our formula from class we get

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

$$= \ln(x) \int \frac{e^{-\int \frac{1}{x} dx}}{(\ln(x))^2} dx$$

$$= \ln(x) \int \frac{e^{-\ln|x|}}{(\ln(x))^2} dx$$

$x > 0$
when
x is in

$$= \ln(x) \int \frac{e^{-\ln(x)}}{(\ln(x))^2}\, dx$$

$I = (0, \infty)$
So, $|x| = x$

$A \ln(B) = \ln(B^A)$

$$= \ln(x) \int \frac{e^{\ln(x^{-1})}}{(\ln(x))^2}\, dx$$

$e^{\ln(z)} = z$

$$= \ln(x) \int \frac{x^{-1}}{(\ln(x))^2}\, dx$$

$$= \ln(x) \int \frac{1}{x(\ln(x))^2}\, dx$$

$\int \frac{1}{x(\ln(x))^2} dx$

$u = \ln(x)$
$du = \frac{1}{x} dx$

$= \int \frac{1}{u^2} du = \int u^{-2} du$

$= \frac{u^{-2+1}}{-2+1} = -u^{-1} = -\frac{1}{\ln(x)}$

$$= \ln(x) \left(-\frac{1}{\ln(x)} \right)$$

$$= -1$$

Thus, $y_1 = x^4$ and $y_2 = -1$ are two linearly independent solutions to $xy'' + y' = 0$ on $I = (0, \infty)$. And the general solution to $xy'' + y' = 0$ on $I = (0, \infty)$ is of the form $y = c_1 y_1 + c_2 y_2 = c_1 \ln(x) + c_2(-1)$

$\boxed{1(d)}$ We are given that $y_1 = x^{1/2}\ln(x)$ is a solution to

$$4x^2 y'' + y = 0$$

on $I = (0, \infty)$.

Note that $y_1 = x^{1/2}\ln(x) \neq 0$ on $I = (0, \infty)$.

Divide by $4x^2$ to get the equation

$$y'' + \frac{1}{4x^2} y = 0$$

there is no y' term so $a_1(x) = 0$

Using our formula from class we get

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

$\int 0 \, dx = 0$

$$= x^{1/2}\ln(x) \int \frac{e^{-\int 0 \, dx}}{(x^{1/2}\ln(x))^2} dx$$

$e^0 = 1$

$$= x^{1/2}\ln(x) \int \frac{e^0}{x(\ln(x))^2} dx$$

$$= x^{1/2}\ln(x)\int \frac{dx}{x(\ln(x))^2}$$

$$= x^{1/2}\ln(x)\left(-\frac{1}{\ln(x)}\right)$$

$$=-x^{1/2}$$

$u=\ln(x)$

$du=\frac{1}{x}dx$

$$\int \frac{1}{x(\ln(x))^2}dx$$

$$=\int \frac{1}{u^2}du = \int u^{-2}du$$

$$=\frac{u^{-2+1}}{-2+1} = -u^{-1} = -\frac{1}{\ln(x)}$$

Thus, $y_1 = x^{1/2}\ln(x)$ and $y_2 = -x^{1/2}$ are two linearly independent solutions to $4x^2y''+y=0$ on $I=(0,\infty)$. And the general solution to the $4x^2y''+y=0$ on $I=(0,\infty)$ is of the form $y=c_1y_1+c_2y_2 = c_1x^{1/2}\ln(x)+c_2(-x^{1/2})$

$\boxed{1(e)}$ We are given that $y_1 = x^{-4}$ is a solution to

$$x^2 y'' - 20y = 0$$

on $I = (0, \infty)$.

Note that $y_1 = x^{-4} \neq 0$ on $I = (0,\infty)$.

Divide by x^2 to get the equation

$$y'' - \frac{20}{x^{-4}} y = 0$$

there is no y' term so $a_1(x) = 0$

Using our formula from class we get

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

$\int 0 \, dx = 0$

$$= x^{-4} \int \frac{e^{-\int 0 \, dx}}{(x^{-4})^2} dx$$

$e^0 = 1$

$$= x^{-4} \int \frac{e^0}{x^{-8}} dx$$

$$= x^{-4} \int \frac{1}{x^{-8}} \, dx$$

$$= x^{-4} \int x^{8} \, dx$$

$$= x^{-4} \frac{x^{9}}{9}$$

$$= \frac{1}{9} x^{5}$$

Thus, $y_1 = x^{-4}$ and $y_2 = \frac{1}{9} x^5$ are two linearly independent solutions to $x^2 y'' - 20y = 0$ on $I = (0, \infty)$. And the general solution to $x^2 y'' - 20y = 0$ on $I = (0, \infty)$ is of the form $y = c_1 y_1 + c_2 y_2 = c_1 x^{-4} + c_2 \left(\frac{1}{9} x^5\right)$

$\boxed{①(f)}$ We are given that $y_1 = e^x$ is a solution to

$$xy'' - (x+1)y' + y = 0$$

on $I = (0, \infty)$.

Note that $y_1 = e^x \neq 0$ on $I = (0, \infty)$.

Divide by x to get the equation

$$y'' \underbrace{- \frac{x+1}{x}}_{a_1(x) = -\frac{x+1}{x}} y' + \frac{1}{x} y = 0$$

Using our formula from class we get

$$y_2 = y_1 \cdot \int \frac{e^{-\int a_1(x)dx}}{y_1^2} dx$$

$$= e^x \int \frac{e^{-\int -\frac{x+1}{x} dx}}{(e^x)^2} dx$$

$$\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$$

$$= e^x \int \frac{e^{\int (1+\frac{1}{x}) dx}}{e^{2x}} dx$$

$$= e^x \int \frac{e^{x+\ln|x|}}{e^{2x}}\,dx$$

$|x| = x$
since $x > 0$
when x is in
$I = (0, \infty)$

$$= e^x \int \frac{e^{x+\ln(x)}}{e^{2x}}\,dx$$

$$= e^x \int \frac{e^x e^{\ln(x)}}{e^{2x}}\,dx$$

$$\frac{e^x}{e^{2x}} = \frac{\cancel{e^x}}{\cancel{e^x} e^x} = \frac{1}{e^x}$$

$$= e^x \int \frac{e^{\ln(x)}}{e^x}\,dx$$

$$e^{\ln(A)} = A$$

$$= e^x \int \frac{x}{e^x}\,dx$$

$$= e^x \int x e^{-x}\,dx$$

$$\int x e^{-x} dx$$

$$= -xe^{-x} + \int e^{-x}$$

$$= -xe^{-x} - e^{-x}$$

$u = x \quad du = dx$

$dv = e^{-x} \quad v = -e^{-x}$

$$= e^x\left(-xe^{-x} - e^{-x}\right)$$

$$= -xe^{x}e^{-x} - e^{x}e^{-x}$$

$$= -xe^{x-x} - e^{x-x}$$

$$= -xe^{0} - e^{0}$$

$$= -x - 1$$

Thus, $y_1 = e^x$ and $y_2 = -x-1$ are two linearly independent solutions to $xy'' - (x+1)y' + y = 0$ on $I = (0, \infty)$. And the general solution to $xy'' - (x+1)y' + y = 0$ on $I = (0, \infty)$ is of the form $y = c_1 y_1 + c_2 y_2 = c_1 e^x + c_2(-x-1)$

② We are given that $y_1 = x^2$ and $y_2 = x^3$ are linearly independent solutions to

$$x^2 y'' - 4x y' + 6y = 0$$

on $I = (0, \infty)$

(a) Let's find a particular solution to

$$x^2 y'' - 4x y' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$

First divide by x^2 to get into standard form:

$$y'' - \frac{4}{x} y' + \frac{6}{x^2} y = \underbrace{\frac{1}{x^3}}_{b(x)}$$

Then

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

$$= 3x^4 - 2x^4$$

$$= x^4$$

Let

$$V_1 = -\int \frac{y_2 \cdot b(x)}{W(y_1, y_2)} = -\int \frac{x^3 \cdot \frac{1}{x^3}}{x^4} dx$$

$$= -\int x^{-4} dx = -\frac{x^{-3}}{-3} = \frac{1}{3} x^{-3}$$

And

$$V_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)} = \int \frac{x^2 \frac{1}{x^3}}{x^4} dx$$

$$= \int \frac{1}{x^5} dx$$

$$= \int x^{-5} dx$$

$$= \frac{x^{-4}}{-4}$$

$$= -\frac{1}{4} x^{-4}$$

Thus, a particular solution to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$ is

$$y_p = v_1 y_1 + v_2 y_2$$

$$= \frac{1}{3} x^{-3} x^2 - \frac{1}{4} x^{-4} x^3 = \frac{1}{3} x^{-1} - \frac{1}{4} x^{-1}$$

$$= \frac{1}{12} x^{-1}$$

(b) The general solution to

$$x^2 y'' - 4xy' + 6y = \frac{1}{x}$$

on $I = (0, \infty)$ is

$$y = y_h + y_p = c_1 x^2 + c_2 x^3 + \frac{1}{12} x^{-1}$$

(3) We are given that $y_1 = x$ and $y_2 = x\ln(x)$ are linearly independent solutions to

$$x^2 y'' - xy' + y = 0$$

on $I = (0, \infty)$

(a) Let's find a particular solution to

$$x^2 y'' - xy' + y = 4x\ln(x)$$

on $I = (0, \infty)$

First divide by x^2 to get into standard form:

$$y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \underbrace{\frac{4}{x}\ln(x)}_{b(x)}$$

Then

$$W(y_1, y_2) = \begin{vmatrix} x & x\ln(x) \\ 1 & \ln(x) + 1 \end{vmatrix}$$

$$= x\ln(x) + x - x\ln(x)$$

$$= x$$

Let

$$V_1 = -\int \frac{y_2 \cdot b(x)}{W(y_1, y_2)} = -\int \frac{x\ln(x) \cdot \frac{4}{x}\ln(x)}{x}\,dx$$

$$= -4\int \frac{(\ln(x))^2}{x}\,dx$$

$$= -\frac{4}{3}(\ln(x))^3$$

$$\int \frac{(\ln(x))^2}{x}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \int u^2\,du$$

$$= \frac{u^3}{3}$$

$$= \frac{(\ln(x))^3}{3}$$

And

$$V_2 = \int \frac{y_1 \cdot b(x)}{W(y_1, y_2)} = \int \frac{x\,\frac{4}{x}\ln(x)}{x}\,dx$$

$$= 4\int \frac{\ln(x)}{x}\,dx$$

$$= 4\left(\frac{1}{2}(\ln(x))^2\right)$$

$$= 2(\ln(x))^2$$

$$\int \frac{\ln(x)}{x}\,dx$$

$u = \ln(x)$

$du = \frac{1}{x}\,dx$

$$= \int u\,du$$

$$= \frac{u^2}{2}$$

$$= \frac{1}{2}(\ln(x))^2$$

Thus, a particular solution to

$$x^2 y'' - xy' + y = 4x\ln(x)$$

on $I = (0, \infty)$ is

$$y_p = v_1 y_1 + v_2 y_2$$

$$= -\frac{4}{3}(\ln(x))^3 \cdot x + 2(\ln(x))^2 \cdot x\ln(x)$$

$$= \frac{2}{3}x(\ln(x))^3$$

(b) The general solution to

$$x^2 y'' - xy' + y = 4x\ln(x)$$

on $I = (0, \infty)$ is

$$y = y_h + y_p = c_1 x + c_2 x\ln(x) + \frac{2}{3}x(\ln(x))^3$$